Test 1 **Theory of Probability** **February 14, 2019**

MATH 464 **Dr. Abdul-Rahman**

Name: ______________________ **Signature:** ______________________

1. [15 points] You have three boxes containing balls of equal size and weight. The first box has 6 yellow balls, 4 red balls, and 2 green balls. The second box has 2 yellow balls, 6 red balls, and 5 green balls. The third box has 5 yellow balls, 5 red balls, and 5 green balls.

   (a) Select a box at random and then select a ball at random from that box. What is the probability that the ball you selected is red?

   (b) Suppose the ball you select is yellow. What is the probability that this ball came from the second box?

2. [10 points] A discrete random variable X has probability mass function f_X defined as follow

$$f_X(k) = \begin{cases} \frac{c}{2^k}, & \text{if } k = 0, 1, 2, \ldots. \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Find the following

(a) the value of c.

(b) $\mathbb{P}(-1 \leq X < 2)$.

(c) $\mathbb{P}(X \text{ is even} \mid X \leq 5)$.

3. [10 points]

   (a) You choose 5 cards from a standard deck of cards. What is the probability that you get 3 red cards and 2 black cards?

   (b) Ten students, 5 boys and 5 girls, lineup in a random order. What is the probability that the boys and girls alternate such that the first student from the right is a girl?

4. [10 points] Distribute k distinguishable balls into n cells at random, multiply occupancy being permitted.

   (a) Find the probability tht the first cell contains exactly m_1 balls.

   (b) Find the probability that the first cell contains exactly m_1 balls and the second cell contains exactly m_2 balls.

5. [10 points] Let A, B, and C be three events such that $A \subseteq B \subseteq C$, and

$$\mathbb{P}(C) = 0.7, \quad \mathbb{P}(C \cap B^c) = 0.2, \quad \mathbb{P}(A) = 0.2.$$

Find the folowing

(a) $\mathbb{P}(B)$.

(b) $\mathbb{P}(B \cap A^c)$.

(c) $\mathbb{P}(A^c \cap C)$.

6. [10 points] Let A and B be any events, show that the probability that exactly one of them occurs is

$$\mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B)$$

7. [10 points] If A and B are independent, prove that A^c and B are independent.

8. [10 points] Let $\mathbb{P}(C) = 0.5$ and $\mathbb{P}(D) = 0.6$, find the maximum and the minimum values of $\mathbb{P}(C \cap D)$.

9. [10 points] Consider an unfair coin which has probability 1/3 for heads and 2/3 for tail. A stubborn person tosses this coin until it lands heads-up.

   (a) Describe the sample space.

   (b) Find the probability it takes at leat 3 tosses.

10. [10 points] A certain rare blood type can be found in only 0.02% of people. Use the Poisson approximation to compute the probability that at most two persons in a group of randomly selected 5000 people will have this rare blood type.

In the following $0 < p < 1$ and $q := 1 - p$.

Bernoulli: $X \sim \mathrm{Bernoulli}(p)$

$$\mathbb{P}(X = 1) = p \quad \text{and} \quad P(X = 0) = q\,.$$

$$\mathbb{E}(X) = p, \quad \mathrm{Var}(X) = pq,\ M_X(t) = q + pe^t.$$

Binomial: $X \sim \mathrm{Binomial}(n, p)$, $n \in \mathbb{N}$.

$$\mathbb{P}(X = k) = \binom{n}{k} q^{n-k} p^k, \quad k = 0, 1, \ldots, n.$$

$$\mathbb{E}(X) = np, \quad \mathrm{Var}(X) = npq,\ M_X(t) = (q + pe^t)^n.$$

Geometric: $X \sim \mathrm{Geometric}(p)$

$$\mathbb{P}(X = k) = q^{k-1} p, \quad k = 1, 2, \ldots$$

$$\mathbb{E}(X) = \frac{1}{p}, \quad \mathrm{Var}(X) = \frac{q}{p^2},\ M_X(t) = \frac{pe^t}{1 - qe^t}.$$

Poisson: $X \sim \mathrm{Poisson}(\lambda)$, $\lambda > 0$.

$$\mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, \ldots$$

$$\mathbb{E}(X) = \lambda, \quad \mathrm{Var}(X) = \lambda,\ M_X(t) = e^{\lambda(e^t - 1)}.$$

Negative Binomial: $X \sim \mathrm{NB}(n, p)$, $n \in \mathbb{N}$.

$$\mathbb{P}(X = k) = \binom{k-1}{n-1} p^n q^{k-n}, \quad k = n, n+1, \ldots$$

$$\mathbb{E}(X) = \frac{n}{p}, \quad \mathrm{Var}(X) = \frac{nq}{p^2},\ M_X(t) = \left(\frac{pe^t}{1 - qe^t}\right)^n.$$
